§4.3 Linear Independence and Bases

We defined these terms for $\mathbb{R}^n$ and now we generalize.

<u>Defn</u>: Let V be a vector space and $v_1, \ldots, v_\ell$ vectors in V. We say $\{v_1, \ldots, v_\ell\}$ is <u>linearly independent</u> if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_\ell v_\ell = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_\ell = 0$. We say this set is <u>linearly dependent</u> if the equation above has a non-trivial solution.

<u>Examples</u>

- In $\mathbb{P}_2$ $\{t^2+1, 3t\}$ is linearly independent
- In $M_{2\times 2}$ $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is linearly independent
- If V is the vector space of real-valued cont. funct. $\{2\sin^2(x), 5\cos^2(x), 3\}$ is linearly dependent!

$$\frac{1}{2} \cdot 2\sin^2(x) + \frac{1}{5} \cdot 5\cos^2(x) + \left(-\frac{1}{3}\right) \cdot 3 = 0$$

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Remarks

Let V be a vector space

1) If v is in V, then $\{v\}$ is linearly independent if and only if $v \neq 0$ zero vector of V

2) $\{u, v\}$ is linearly independent if and only if u and v are not multiples of each other. (This only works for sets of 2 vectors!)

3) Any set containing the zero vector is linearly dependent.

4) A set $\{v_1, \ldots, v_\ell\}$ $\ell \geq 2$ is linearly dependent if and only if at least one of the vectors is a linear combination of the others, i.e.

some v_i is in $\text{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_\ell\}$

for some i.

<u>Defn</u>: Let V be a vector space and H a subspace. A set $\mathcal{B} = \{u_1, \ldots, u_\ell\}$ of vectors in H is a <u>basis</u> of H if

1) $\mathcal{B} = \{u_1, \ldots, u_\ell\}$ is linearly independent
2) $H = \text{span } \mathcal{B} = \text{span}\{u_1, \ldots, u_\ell\}$

<u>Examples</u>

1) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is the <u>standard basis</u> for $\mathbb{R}^3$

in general, the columns of I_n are the standard basis of $\mathbb{R}^n$

2) $\{1, t, t^2, \ldots, t^n\}$ is the standard basis of $\mathbb{P}_n$

3) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis of $M_{2\times 2}$

A basis is the most "efficient" spanning set in a sense. Any spanning set can be reduced to a basis by omitting any vectors that are linear combinations of the others.

For example, let

$$v_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

and let $H = \text{span}\{v_1, v_2, v_3\}$, which is a subspace of $\mathbb{R}^3$. Notice that

$$v_2 = v_1 + 2v_3$$

so $H = \text{span}\{v_1, \cancel{v_2}, v_3\} = \text{span}\{v_1, v_3\}$

Thus $\{v_1, v_3\}$ is a basis for H since this set is linearly independent.

Notice, we could have also said $v_1 = v_2 - 2v_3$

so $H = \text{span}\{\cancel{v_1}, v_2, v_3\} = \text{span}\{v_2, v_3\}$

so $\{v_2, v_3\}$ is also a basis since this set is linearly independent.

Recall if A is an $m \times n$ matrix, the column space of A is a subspace of $\mathbb{R}^m$ and the pivot columns of A form a basis for Col A.

What about the rows of A?

<u>Defn</u>: If A is an $m \times n$ matrix, the <u>row space</u> of A is the subspace of $\mathcal{M}_{1 \times n}$ (set of $1 \times n$ matrices) which is spanned by the rows of A. We denote it by <u>Row A</u>.

Remark: Sometimes, for convenience, we write these rows vertically. Thus by taking transposes, we may identify Row A with a subspace of $\mathbb{R}^n$.

- Notice that row operations don't change the row space!
- They do change the column space however.

Theorem

Let A be a matrix and E any echelon form of A (not necessarily reduced).

1) The pivot columns of A (not E!) form a basis for Col A.

2) The nonzero columns of E form a basis for Row A

Example

Find Bases for Col A and Row A if

$$A = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 4 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

doesn't have to be reduced echelon form!

$$A \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1) Basis for Col A: $\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \right\}$ Pivot columns of A

2) Basis for Row A:

$$\left\{ [1\ 2\ 0\ 0], [0\ 0\ 1\ 0], [0\ 0\ 0\ 5] \right\}$$

nonzero rows of an echelon form

Exercises

1) A square matrix A is <u>symmetric</u> if $A = A^T$. Find a basis for the subspace of $M_{2\times2}$ consisting of all 2×2 symmetric matrices.

2) A square matrix A is <u>skew-symmetric</u> if $A = -A^T$. Find a basis for the subspace of $M_{2\times2}$ consisting of all 2×2 skew-symmetric matrices.

3) Let H be the subspace of $\mathbb{P}_2$ consisting of all polynomials of degree at most 2 such that $p(1) = p(0)$.
   a) show H is a subspace of $\mathbb{P}_2$
   b) Find a basis for H.

4) Let H be the subspace of $\mathbb{P}_3$ consisting of all polynomials $p(t)$ of degree at most 3 satisfying $p(1) = p(0)$. Find a basis of H.

5) Do the same thing as in #4, but with the subspace of $\deg \le 3$ polynomials $p(t)$ such that $p(-1) = p(1)$.